To the Reader:

Although the following, part of my relentless effort to find groups in everything, is, I believe, rigorously correct, it was written a bit tongue-in-cheek just to see if I could render an otherwise fascinating topic entirely sterile with my best math journalistic prose.

Why Pianos are Always Out of Tune

Let $N_0, N_1, N_2, \ldots$ denote the names of notes, and let $n_0, n_1, n_2, \ldots$ denote their corresponding frequencies. These notes are ordered such that $n_0 < n_1 < n_2 < \ldots$.

Two notes N_i and N_j form an interval of one *octave* if the ratio of their frequencies $n_i/n_j = 2$ or $n_i/n_j = \frac{1}{2}$. Thus two notes form an interval of k octaves if $n_i/n_j = 2^k$ or $n_i/n_j = 2^{-k}$.

The *ascending chromatic scale* is defined as a set of twelve notes $\{N_i, N_{i+1}, N_{i+2}, \ldots, N_{i+11}\}$ such that $n_{i+12}/n_i = 2$.

Two notes N_i and N_j form an *interval of a fifth* if $|i - j| = 7$. An interval of a fifth is perfectly tuned if the ratio of frequencies $n_i/n_j = \frac{3}{2}$ or $n_i/n_j = \frac{2}{3}$.

We define an equivalence "=" of notes: $N_i = N_j$ whenever $n_i/n_j = 2^k$ or $n_i/n_j = 2^{-k}$. This "=" is easily verified to be an equivalence relation.

We define a binary operation "•" as follows: $N_i \bullet N_j = N_{i+j}$.

It can be seen, then, that the set $\{N_i, N_{i+1}, N_{i+2}, \ldots, N_{i+11}\}$ forms the cyclic group of order twelve under "•". The identity, N_i, is known as the *tonic*.

Two necessary conditions for a piano to be perfectly tuned are that any interval of a fifth is perfectly tuned and that any subset of twelve notes on a piano $\{N_i, N_{i+1}, N_{i+2}, \ldots, N_{i+11}\}$ forms an ascending chromatic scale.

Theorem: A piano cannot be perfectly tuned.

Proof: Suppose it could be perfectly tuned. Since N_i and N_{i+7} form an interval of a fifth, $n_{i+7} / n_i = \frac{3}{2}$. Then $n_{i+14} / n_i = \left(\frac{3}{2}\right)^2$, and so on, such that $n_{i+84} / n_i = \left(\frac{3}{2}\right)^{12}$. But $N_{i+84} = (N_{i+7})^{12} = N_i$ since N_{i+7} has order twelve. This would imply that $n_{i+84} / n_i = 2^k$, a contradiction.

J. Davidson, Feb., 1993